Lambek–Grishin Calculus Extended to Connectives of Arbitrary Arity

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Abstract

This paper introduces Lambek–Grishin calculus for n-ary connectives, which can be seen as a generalization of binary and unary Lambek–Grishin calculus. It is a categorial grammar that is at least mildly context-sensitive, and allows for branching of arbitrary arity. Therefore it seems very suitable for linguistic applications. A cut-free presentation of the calculus is showed, proving the decidability of the calculus. Finally we investigate the symmetries of the calculus by making use of group theory.

1 Introduction

One of the disadvantages of traditional categorial grammars like standard Lambek calculus [8] and its symmetric extension, the binary Lambek–Grishin calculus [9], is that they only allow for binary branching. However, in many linguistic examples it seems to be more natural to also admit ternary branching or branching of even higher order. Consider for example sentences with ditransitive verbs like ′He gives the man a book′. We might want to see the phrase ′gives the man a book′ as a branch with three direct constituents, namely ′gives′, ′the man′ and ′a book′.

One of the solutions to this problem is the extension of Lambek calculus to an n-ary variant. For example Buszkowski [2] proposed a system with residuation modalities of arbitrary arity, but in [6] it was proved that this calculus recognizes only the context-free languages. Therefore this formalism might not be a good choice for natural language, as it has been argued that natural language is not context-free [5].

In this paper, we present a categorial grammar that has both advantages: it is (at least) mildly context-sensitive, and still we have the possibility of describing branching or arbitrary arity. The calculus can be seen as a generalization of the binary Lambek–Grishin calculus [9] and the unary Lambek–Grishin calculus [3] to arbitrary arity. Binary Lambek–Grishin calculus is mildly context-sensitive (and parsable in polynomial time) [11].

In the next section, I will present the Lambek–Grishin calculus for unary and binary connectives. In section 3, I introduce the Lambek–Grishin calculus for n-ary arity. In section 4, I show that this calculus is decidable. In section 5 I study the symmetries of the calculus with aid of group theory. Finally I draw some conclusions in section 6, and point out some lines for further research.

2 Lambek–Grishin calculus for binary and unary connectives

2.1 The binary Lambek–Grishin calculus

Definition 1 (Minimal Lambek–Grishin calculus)**.** The *Lambek–Grishin calculus* is an extension of the *nonassociative Lambek calculus* [8]. Lambek–Grishin calculus is, just like the ordinary Lambek calculus, a deductive system. A sentence is grammatical if and only if the sentence can be derived in the deductive system. To every word in the language a finite number of types is assigned. We have a set of basic types called *Atoms*. The set of *types* T is defined as follows: every atom is a type, and if a and b are types, then the following formulas are types as well:

$$a \otimes b, \quad a\backslash b, \quad a/b, \quad a \oplus b, \quad a \obslash b, \quad a \oslash b.$$

Intuitively we can see the $\otimes$-operator as concatenation of words. Furthermore one or more words with type $a\backslash b$ can be seen as a group of words which need a phrase of type a on their left, and return a phrase of type b. In the same way we can see a phrase of type a/b as a phrase which asks for a phrase of type b on its right-hand side, before returning an a type phrase. Furthermore we can see $\oplus$, $\oslash$ and $\oslash$ as the duals of $\otimes$, $/$ and $\backslash$ under arrow reversal. We define the *size* of a type a, denoted $|a|$, as the number of connectives it contains.

Minimal Lambek–Grishin calculus [9] for binary connectives $\mathbf{LG_2^\emptyset}$ is defined as the calculus having the following rules:

$$a \to a \qquad \frac{a \to b \qquad b \to c}{a \to c}\text{ Cut} \qquad \cfrac{\cfrac{b \to a\backslash c}{a \otimes b \to c}\text{ Res}}{a \to c/b}\text{ Res} \qquad \cfrac{\cfrac{a \oslash c \to b}{c \to a \oplus b}\text{ Res}}{c \oslash b \to a}\text{ Res}$$

From left to right: *identity*, *transitivity* (also named '*cut*'), and *residuation* (2x). With the double line we indicate derivability in both directions. In the cut rule, we call b the *cut formula*. The *complexity* of a cut is defined as $|a| + |b| + |c|$.

Definition 2 (Lambek–Grishin calculus with interaction)**.** Grishin [4] proposes four classes of postulates to govern the interaction between the $\oplus$ and $\otimes$ connectives. The Lambek–Grishin calculus $\mathbf{LG}$, also notated as $\mathbf{LG_2^\emptyset} + \mathbf{IV}$, is the extension of the minimal calculus $\mathbf{LG_2^\emptyset}$ with the postulates from Grishin class IV added.

$$\frac{a \oslash (b \otimes c) \to d}{(a \oslash b) \otimes c \to d}\, G1 \quad \frac{b \oslash (a \otimes c) \to d}{a \otimes (b \oslash c) \to d}\, G2 \quad \frac{(a \otimes b) \oslash c \to d}{a \otimes (b \oslash c) \to d}\, G3 \quad \frac{(a \otimes c) \oslash b \to d}{(a \oslash b) \otimes c \to d}\, G4$$

Definition 3 (Lambek grammar)**.** A *Lambek grammar* has the form $\mathcal{L}(\Sigma, s, f)$, such that for all $t \in \Sigma$, $f(t) \subset T$ and $f(t)$ is finite, where T is the set of all types of the Lambek calculus. We define the *yield* of a formula consisting of only $\otimes$-connectives as follows: $\text{yield}(a) = a$ if a is an atom, $\text{yield}(a \otimes b) = \text{yield}(a)\text{yield}(b)$. The *language generated by the Lambek grammar* $\mathcal{L}(\Sigma, s, f)$ is defined as the set of all expressions $t_1 \dots t_n$ over the alphabet Σ for which there exists a derivable sequent with yield $b_1 \dots b_n \to s$ such that $b_i \in f(t_i)$ for all $i \leq n$.

2.2 Symmetries

Definition 4 (Order-preserving symmetry)**.** We define on formulas the operations $\bowtie$, $\sharp$ and $\flat$. The operation $\bowtie$ exchanges the order of all types, while $\sharp$ and $\flat$ only exchange the order of the $\otimes$- and $\oplus$-family connectives, respectively. Formally the operations are defined as follows:

$$\begin{array}{rcl@{\qquad}rcl@{\qquad}rcl}
a^{\bowtie} &=& a \quad \text{if } a \in Atoms & a^{\sharp} &=& a \quad \text{if } a \in Atoms & a^{\flat} &=& a \quad \text{if } a \in Atoms \\
(a \otimes b)^{\bowtie} &=& b^{\bowtie} \otimes a^{\bowtie} & (a \otimes b)^{\sharp} &=& b^{\sharp} \otimes a^{\sharp} & (a \otimes b)^{\flat} &=& a^{\flat} \otimes b^{\flat} \\
(a/b)^{\bowtie} &=& b^{\bowtie}\backslash a^{\bowtie} & (a/b)^{\sharp} &=& b^{\sharp}\backslash a^{\sharp} & (a/b)^{\flat} &=& a^{\flat}/b^{\flat} \\
(a\backslash b)^{\bowtie} &=& b^{\bowtie}/a^{\bowtie} & (a\backslash b)^{\sharp} &=& b^{\sharp}/a^{\sharp} & (a\backslash b)^{\flat} &=& a^{\flat}\backslash b^{\flat} \\
(a \oplus b)^{\bowtie} &=& b^{\bowtie} \oplus a^{\bowtie} & (a \oplus b)^{\sharp} &=& a^{\sharp} \oplus b^{\sharp} & (a \oplus b)^{\flat} &=& b^{\flat} \oplus a^{\flat} \\
(a \oslash b)^{\bowtie} &=& b^{\bowtie} \oslash a^{\bowtie} & (a \oslash b)^{\sharp} &=& a^{\sharp} \oslash (b) & (a \oslash b)^{\flat} &=& b^{\flat} \oslash (a) \\
(a \oslash b) &=& b^{\bowtie} \oslash a^{\bowtie} & (a \oslash b)^{\sharp} &=& a^{\sharp} \oslash b^{\sharp} & (a \oslash b)^{\flat} &=& b^{\flat} \oslash a^{\flat}
\end{array}$$

Theorem 5 ([9])**.** *It holds that* $a^{\bowtie} \to b^{\bowtie}$ *if and only if* $a \to b$.

Theorem 6 ([9])**.** *The operations* $\bowtie$, $\sharp$ *and* $\flat$ *form together with the identity transformation a group under function composition isomorphic to* $\mathbb{Z}_2 \times \mathbb{Z}_2$.

Definition 7 (Order-reversing symmetry)**.** Another operator is ∞, with the following definition. We have $a^{\infty} = a$ whenever a is an atom. Furthermore we define

$$\begin{array}{rclrcl}(a\otimes b)^\infty &=& b^\infty \oplus a^\infty; & (a\oplus b)^\infty &=& b^\infty \otimes a^\infty;\\ (a/b)^\infty &=& b^\infty \obslash a^\infty; & (a\oslash b)^\infty &=& b^\infty \backslash a^\infty;\\ (a\backslash b)^\infty &=& b^\infty \oslash a^\infty; & (a\obslash b)^\infty &=& b^\infty / a^\infty.\end{array}$$

Theorem 8 ([9])**.** *It holds that $a^\infty \to b^\infty$ if and only if $b \to a$.*

Theorem 9 ([9])**.** *The group generated by ∞, $\flat$ and $\sharp$ is isomorphic to D_4, the dihedral group with 8 elements.*

2.3 The unary Lambek–Grishin calculus

Definition 10. Lambek–Grishin calculus with unary connectives was proposed in [3]. The minimal unary Lambek–Grishin calculus $\mathbf{LG_1^\emptyset}$ contains the following types: every atom is a type, and if a is a type, then $\diamond_1 a$, $\square_1 a$, $\diamond_2 a$ and $\square_2 a$ are types as well. Again we have the axiom $a \to a$ and transitivity. The residuation rules for the unary calculus are defined in the following way:

$$\forall i \in \{1,2\} \quad \diamond_i a \to b \text{ iff } a \to \square_i b.$$

Just like for the binary variant, we can add interaction postulates, which gives us $\mathbf{LG_1^\emptyset + IV}$. In the unary case, we have only one interaction postulate (appearing in three interderivable forms, though). Therefore we have only the trivial group with one element. One way of writing down this interaction postulate is the following:

$$\diamond_1\diamond_2 a \to \diamond_2\diamond_1 a.$$

3 Lambek–Grishin calculus for n-ary connectives

Now we have seen the binary and unary Lambek–Grishin calculus, we can extend the Lambek–Grishin calculus to arbitrary arity.

Definition 11. We define the types for n-ary Lambek–Grishin calculus $\mathbf{LG_n^\emptyset}$ as follows. Again we define a set of atoms. Every atom is a type, and if a_k is a type for every k such that $1 \le k \le n$ then $f_\bullet(a_1,\dots,a_n)$, $f^i_\to(a_1,\dots,a_n)$, $g_\bullet(a_1,\dots,a_n)$ and $g^i_\to(a_1,\dots,a_n)$ are types for every i such that $1 \le i \le n$. We should see f and g as galois connected pairs, i.e. $f_\bullet(a_0,\dots,a_n) \to b$ if and only if $b \to g_\bullet(a_0,\dots,a_n)$, just like the relation between $\otimes$ and $\oplus$ in the binary calculus. Furthermore we should see $f_\to$ and $g_\to$ as the residue operators of $f_\bullet$ and $g_\bullet$, respectively. From a (linear) logical perspective, we can see the $\bullet$-connectives $f_\bullet$ and $g_\bullet$ as multiplication, while we can see the $\to$-connectives $f_\to$ and $g_\to$ as implication. The superscript i in the implications denotes which element is the goal type of the implication. We can see it as succedent or conclusion of the implication, while the other $n-1$ elements can be seen as antecedents.

Remark 12. Of course, the connectives of binary and unary Lambek–Grishin calculus can be expressed in this notation as well. For binary Lambek–Grishin calculus we have the following correspondences:

$a\otimes b$	a/b	$a\backslash b$	$a\oplus b$	$a\oslash b$	$a\obslash b$
$f_\bullet(a,b)$	$f^1_\to(a,b)$	$f^2_\to(a,b)$	$g_\bullet(a,b)$	$g^1_\to(a,b)$	$f^2_\to(a,b)$

For unary Lambek–Grishin calculus, the correspondences are as follows:

$\diamond_1$	$\square_1$	$\square_2$	$\diamond_2$
$f_\bullet(a)$	$f^1_\to(a)$	$g_\bullet(a)$	$g^1_\to(a)$

The minimal Lambek–Grishin calculus for n-ary connectives $\mathbf{LG_n^\emptyset}$ has the following axioms and rules:

$$a \to a \quad \text{(identity)} \tag{1}$$

$$\frac{a \to b \qquad b \to c}{a \to c}\text{Cut} \tag{2}$$

Residuation rules:

$$\frac{f_\bullet(a_1,\ldots,a_n)\to b}{a_i\to f^i_\to(a_1,\ldots,a_{i-1},b,a_{i+1},\ldots,a_n)}\text{ Res} \quad (3)$$

$$\frac{g^i_\to(a_1,\ldots,a_{i-1},b,a_{i+1},\ldots,a_n)\to a_i}{b\to g_\bullet(a_1,\ldots,a_n)}\text{ Res} \quad (4)$$

One can check easily that the rules for binary and unary Lambek–Grishin calculus are instances of these rules.

Definition 13. The *grammar* belonging to $\mathbf{LG}^{\emptyset}_{\mathbf{n}}+\mathbf{IV}$ can be defined in the same way as the grammar for $\mathbf{LG}^{\emptyset}_{\mathbf{2}}+\mathbf{IV}$ as defined in definition 3. However, the definition of yield will be adapted, such that the *yield* of a formula consisting of only $f_\bullet$ connectives is defined as follows: $\text{yield}(a)=a$ if a is an atom, $\text{yield}(f_\bullet(a_0,\ldots,a_n))=\text{yield}(a_0)\ldots\text{yield}(a_n)$.

3.1 Order-preserving symmetry

Now we will study the symmetries of $\mathbf{LG}^{\emptyset}_{\mathbf{n}}+\mathbf{IV}$. To do this, we define for all permutations π_f,π_g on $(1,\ldots,n)$ a *permutation on formulas* $Z_{\pi_f\pi_g}$:

$$\begin{aligned}
Z_{\pi_f\pi_g}(a) &= a \quad \text{if } a \text{ is an atom;}\\
Z_{\pi_f\pi_g}(f_\bullet(a_1,\ldots,a_n)) &= f_\bullet(Z_{\pi_f\pi_g}(a_{\pi_f(1)}),\ldots,Z_{\pi_f\pi_g}(a_{\pi_f(n)}));\\
Z_{\pi_f\pi_g}(g_\bullet(a_1,\ldots,a_n)) &= g_\bullet(Z_{\pi_f\pi_g}(a_{\pi_g(1)}),\ldots,Z_{\pi_f\pi_g}(a_{\pi_g(n)}));\\
Z_{\pi_f\pi_g}(f^k_\to(a_1,\ldots,a_n)) &= g^{\pi_f(k)}_\to(Z_{\pi_f\pi_g}(a_{\pi_f(1)}),\ldots,Z_{\pi_f\pi_g}(a_{\pi_f(n)}));\\
Z_{\pi_f\pi_g}(g^k_\to(a_1,\ldots,a_n)) &= g^{\pi_g(k)}_\to(Z_{\pi_f\pi_g}(a_{\pi_g(1)}),\ldots,Z_{\pi_f\pi_g}(a_{\pi_g(n)})).
\end{aligned}$$

That is, we apply the π_f permutation to the f-family, and the π_g permutation to the g-family of connectives.

Theorem 14. *It holds that $Z_{\pi_f\pi_g}(a)\to Z_{\pi_f\pi_g}(b)$ if and only if $a\to b$ for all permutations π_f,π_g on $1,\ldots,n$.*

Proof sketch. All axioms and rules of $\mathbf{LG}^{\emptyset}_{\mathbf{n}}$ are closed under permutation on formulas. □

Now we define a rotation operator $\bowtie(i,j)$ as Z_{π_f,π_g}, where $\pi_f(k)=k+i \mod n$ and $\pi_g(k)=k+j \mod n$. This operator rotates the arguments of connectives of the f-family i places, and of the g-family j places. This definition is an extension of the operators $\bowtie$, $\flat$ and $\sharp$ for binary Lambek–Grishin calculus in the following way, for $n=2$: $\bowtie=\bowtie(1,1)$, $\sharp=\bowtie(1,0)$ and $\flat=\bowtie(0,1)$.

3.2 Order-reversing symmetry

We can extend the symmetry function ∞ to our n-ary system:

$$\begin{aligned}
a^\infty &= a \quad \text{if } a \text{ is an atom;}\\
f_\bullet(a_1,\ldots,a_n)^\infty &= g_\bullet(a_n^\infty,\ldots,a_1^\infty);\\
g_\bullet(a_1,\ldots,a_n)^\infty &= f_\bullet(a_n^\infty,\ldots,a_1^\infty);\\
f^i_\to(a_1,\ldots,a_n)^\infty &= g^{n-i}_\to(a_n^\infty,\ldots,a_1^\infty);\\
g^i_\to(a_1,\ldots,a_n)^\infty &= f^{n-i}_\to(a_n^\infty,\ldots,a_1^\infty).
\end{aligned}$$

Theorem 15. *It holds that $a^\infty\to b^\infty$ if and only if $b\to a$.*

3.3 Interaction principles

We proceed by giving postulates which govern interaction between the f- and g-families. An example of such a postulate is

$$f_\bullet(a_1,\dots,a_{n-1},g^1_\rightarrow(b_1,\dots,b_n)) \rightarrow g^1_\rightarrow(f_\bullet(a_1,\dots,a_{n-1},b_1),b_2,\dots,b_n).$$

We obtain the full class of postulates by taking the closure of the given postulate under $\bowtie(i,j)$ such that $0 \leq i \leq n-1$ and $0 \leq j \leq n-1$. This gives us the following postulates, parameterized by i and j:

$$\begin{aligned} & f_\bullet(a_1,\dots,a_{i-1},g^j_\rightarrow(b_1,\dots,b_n),a_{i+1},\dots,a_n) \\ \rightarrow\ & g^j_\rightarrow(b_1,\dots,b_{j-1},f_\bullet(a_1,\dots,a_{i-1},b_j,a_{i+1},\dots,a_n),b_{j+1},\dots,b_n). \end{aligned} \tag{5}$$

Now we can see that this class contains exactly n^2 different postulates (those who are familiar with Grishin's original paper may note that we only considered Grishin class IV). The interaction principles defined in this way are generalizations of the interaction principles for both binary and unary Lambek–Grishin calculus. This explains also why we have four non-equivalent interaction principles per class in binary Lambek–Grishin calculus, while we have only one in unary Lambek–Grishin calculus.

4 Decidability of $\mathbf{LG}^\emptyset_\mathbf{n} + \mathbf{IV}$

Theorem 16. *Lambek–Grishin calculus for arbitrary modalities* $\mathbf{LG}^\emptyset_\mathbf{n} + \mathbf{IV}$ *is decidable.*

Proof. The decidability proof for n-ary Lambek–Grishin calculus is an extension to the decidability proof for binary Lambek–Grishin calculus [9], which in turn is an extension to the decidability proof for Lambek calculus [10]. Note however that we simplified these proofs somewhat, because we do not need the principal cases and the special cases for Grishin interactions.

The strategy for this proof is as follows. We introduce a new axiomatization for $\mathbf{LG}^\emptyset_\mathbf{n} + \mathbf{IV}$, which we prove to be equivalent to out original axiomatization. Next, we prove that in the new axiomatization, the transitivity rule can be eliminated.

Now we will describe the new axiomatization. We keep identity (1), transitivity (2) and the residuation rules (3, 4). However we replace the Grishin interactions by rule-based versions (6), and we add monotonicity (7, 8, 9, 10). The rule-based Grishin interactions look as follows:

For all $1 \leq i \leq n$, $1 \leq j \leq n$ we have:

$$\frac{g^i_\rightarrow(a_1,\dots,a_{i-1},f_\bullet(b_1,\dots,b_n),a_{i+1},\dots,a_n) \rightarrow d}{f_\bullet(b_1,\dots,b_{j-1},g^i_\rightarrow(a_1,\dots,a_{i-1},b_j,a_{i+1},\dots,a_n),b_{j+1},\dots,b_n) \rightarrow d}\ \text{Gr} \tag{6}$$

We add the following monotonicity rules:

$$\frac{a_i \rightarrow b_i \text{ for all } i}{f_\bullet(a_1,\dots,a_n) \rightarrow f_\bullet(b_1,\dots,b_n)}\ \text{Mon} \tag{7}$$

$$\frac{a_k \rightarrow b_k \text{ for } k \neq i \quad b_i \rightarrow a_i}{f^i_\rightarrow(b_1,\dots,b_n) \rightarrow f^i_\rightarrow(a_1,\dots,a_n)}\ \text{Mon} \tag{8}$$

$$\frac{a_i \rightarrow b_i \text{ for all } i}{g_\bullet(a_1,\dots,a_n) \rightarrow g_\bullet(b_1,\dots,b_n)}\ \text{Mon} \tag{9}$$

$$\frac{a_k \rightarrow b_k \text{ for } k \neq i \quad b_i \rightarrow a_i}{g^i_\rightarrow(b_1,\dots,b_n) \rightarrow g^i_\rightarrow(a_1,\dots,a_n)}\ \text{Mon} \tag{10}$$

The old system, consisting of identity, transitivity, residuation, Grishin-interactions is equivalent to the new system, consisting of identity, transitivity, residuation, monotonicity and rule-based Grishin interactions. Grishin interactions and rule-based Grishin interactions can simply be derived from each other with use of transitivity. Now we show how to derive monotonicity. First note that from $a_i \rightarrow b_i$ for all i and $f_\bullet(b_1,\dots,b_n) \rightarrow c$ we can derive $f_\bullet(a_1,\dots,a_n) \rightarrow c$, by iteratively applying the following:

$$\dfrac{a_i \to b_i \qquad \dfrac{f_\bullet(a_1,\ldots,a_{i-1},b_i,b_{i+1},\ldots,b_n) \to c}{b_i \to f^i_\to(a_1,\ldots,a_{i-1},c,a_{i+1},\ldots,a_n)}\,\text{Res}}{\dfrac{a_i \to f^i_\to(a_1,\ldots,a_{i-1},c,a_{i+1},\ldots,a_n)}{f_\bullet(a_1,\ldots,a_{i-1},a_i,b_{i+1}\ldots,b_n) \to c}\,\text{Res}}\,\text{Cut}$$

We call this procedure $\mathbf{P}_\bullet$. In a similar way we can derive from $b_k \to a_k$ for $k \neq i$, $a_i \to b_i$ and $c \to f^i_\to(a_1,\ldots,a_n)$ the formula $c \to f^i_\to(b_1,\ldots,b_n)$. This procedure will be called $\mathbf{P}_\to$. From $\mathbf{P}_\bullet$ and $\mathbf{P}_\to$, we can easily obtain monotonicity.

Now we will show that we can eliminate transitivity (or cut) from the calculus consisting of identity (1), transitivity (2), residuation (3, 4), monotonicity (7, 8, 9, 10) and rule-based Grishin interactions 6. The resulting calculus has no rule with more (instances of) atoms in the hypotheses than in the conclusion, so it is decidable.

First we label every subformula with a polarity. We label $a \to b$ as $a^- \to b^+$. Furthermore, in positive context we have:

$$f_\bullet(a_1^+,\ldots,a_n^+); \quad f^i_\to(a_1^-,\ldots,a_{i-1}^-,a_i^+,a_{i+1}^-,\ldots,a_n^-);$$
$$g_\bullet(a_1^+,\ldots,a_n^+); \quad g^i_\to(a_1^-,\ldots,a_{i-1}^-,a_i^+,a_{i+1}^-,\ldots,a_n^-).$$

In negative context all polarities are reversed. Note that residuation and the Grishin rules do not change the polarity of any subformula.

We show that the highest cut in the proof always can be eliminated, or replaced by n cuts with lower complexity, measured as $|a| + |b| + |c|$. We show this by case analysis on the cut formula of the uppermost cut.

Case 1: The cut formula of the uppermost cut is an atom. In this case, the antecedent of the right premise must be introduced by identity somewhere higher in the proof. Therefore the proof has the form on the left-hand side. As we can replace atoms by any formula, we can replace atom b by formula a, and rewrite the proof as follows:

$$\dfrac{a \to b \qquad \dfrac{\dfrac{b \to b}{\vdots}}{b \to c}}{a \to c} \;\leadsto\; \dfrac{\dfrac{a \to b}{\vdots}}{a \to c}$$

This eliminates the uppermost cut.

Case 2a: The cut formula of the uppermost cut has the form $f_\bullet(a_1,\ldots,a_n)$. As the $f_\bullet$-connective can only be introduced in a positive context by monotonicity, the succedent of the left premise must be introduced by monotonicity somewhere higher in the proof. Therefore the proof has the form on the top. Because no rule other than monotonicity introduces or eliminates $f_\bullet$ in a positive context, we know that $f_\bullet(a_1,\ldots,a_n)$ stays in tact in the proof part represented by the dots, so we can replace it by any formula. Therefore we can rewrite the proof in the following way:

$$\dfrac{\dfrac{\dfrac{\dfrac{a'_k \to a_k \text{ for } k \neq i \qquad a_i \to a'_i}{f^i_\to(a'_1,\ldots,a'_n) \to f^i_\to(a_1,\ldots,a_n)}\,\text{Mon}}{\vdots}}{b \to f^i_\to(a_1,\ldots,a_n)} \qquad f^i_\to(a_1,\ldots,a_n) \to c}{b \to c}\,\text{Cut}$$

$$\leadsto \quad \dfrac{\dfrac{\dfrac{a'_k \to a_k \text{ for all } k \qquad f^i_\to(a_1,\ldots,a_n) \to c}{f^i_\to(a'_1,\ldots,a'_n) \to c}\,P_\bullet}{\vdots}}{b \to c}$$

We eliminate a cut of complexity $|b| + (\sum_{1\leq i\leq n} |a_i| + 1) + |c|$, and add n cuts of maximal complexity $\max_{1\leq k\leq n}(\sum_{1\leq i\leq k} |a_i| + \sum_{k\leq i\leq n} |a'_i|) + |c|$. Because no rule (except cut, which does not occur in the dotted part) has more atoms in one of the hypotheses than in the conclusion, it holds that $\sum_{1\leq i\leq n} |a'_i| \leq |b|$. Therefore the new cuts have indeed lower complexity.

Case 2b: The cut formula of the uppermost cut has the form $f^i_\to(a_1,\dots,a_n)$. As the $f_\to$-connective can only be introduced in a negative context by monotonicity, the antecedent of the right premise must be introduced by monotonicity somewhere higher in the proof. Therefore the proof has the form on the top. Because no rule other than monotonicity introduces $f_\to$ in a negative context, we know that $f^i_\to(a_1,\dots,a_n)$ stays in tact in the proof part represented by the dots. Therefore we can replace it by any formula, so we can rewrite the proof as follows:

$$\dfrac{b \to f^i_\to(a_1,\dots,a_n) \qquad \dfrac{\dfrac{\dfrac{a'_k \to a_k \text{ for } k \neq i \qquad a_i \to a'_i}{f^i_\to(a_1,\dots,a_n) \to f^i_\to(a'_1,\dots,a'_n)}\,\text{Mon}}{\vdots}}{f^i_\to(a_1,\dots,a_n) \to c}}{b \to c}\,\text{Cut}$$

$$\rightsquigarrow \qquad \dfrac{\dfrac{\dfrac{a'_k \to a_k \text{ for } k \neq i \qquad a_i \to a'_i \qquad b \to f^i_\to(a_1,\dots,a_n)}{b \to f^i_\to(a'_1,\dots,a'_n)}\,P_\to}{\vdots}}{b \to c}$$

Again we can check that the new cuts have lower complexity.

Case 2c-d: The cut formula of the uppermost cut has the form $g_\bullet(a_1,\dots,a_n)$ or $g_\to(a_1,\dots,a_n)$. Those cases are symmetric under ∞. Note in particular that $g_\bullet$ can never be introduced or eliminated by Grishin rules at all, and $g_\to$ can never be introduced or eliminated by Grishin rules in a positive context.

$\square$

5 Group-theoretic properties

Now let us consider the symmetries of our system from a group-theoretic perspective, and try to extend theorems 5, 6, 8 and 9. Let us first consider the group with the elements $\bowtie(i,j)$ such that $0 \leq i \leq n-1$ and $0 \leq j \leq n-1$, and function composition as operation. To show that this is indeed a group, we check the required properties. We can easily see that $\bowtie(i,j) \circ \bowtie(i',j') = \bowtie(i+i' \mod n, j+j' \mod n)$, so the set is closed under the group operator. Furthermore composition is associative. The element $\bowtie(0,0)$ functions as identity element, because it is equal to $Z_{I,I}$ where I is the identity permutation. Finally the inverse of $\bowtie(i,j)$ is $\bowtie(-i,-j)$, as $\bowtie(i,j) \circ \bowtie(-i,-j) = \bowtie(-i,-j) \circ \bowtie(i,j) = \bowtie(0,0)$. It turns out that the group in question is isomorphic to $\mathbb{Z}_n \times \mathbb{Z}_n$, with $\bowtie(1,0)$ and $\bowtie(0,1)$ as generators. This is indeed a generalization of the group $\mathbb{Z}_2 \times \mathbb{Z}_2$ that we found for $n = 2$ in theorem 6.

With $\infty\bowtie(i,j)$ we mean the operation consisting of first applying $\bowtie(i,j)$, and then applying ∞. Now we consider the group with the elements $\bowtie(i,j)$ and $\infty\bowtie(i,j)$ such that $0 \leq i \leq n-1$ and $0 \leq j \leq n-1$ and composition as operation. We will abbreviate $\infty\bowtie(0,0)$ with ∞. We have closure because $\bowtie(i,j) \circ \bowtie(i',j') = \bowtie(i+i' \mod n, j+j' \mod n)$, $\infty\bowtie(i,j) \circ \bowtie(i',j') = \infty\bowtie(i+i' \mod n, j+j' \mod n)$, $\bowtie(i,j) \circ \infty\bowtie(i',j') = \infty\bowtie(i'+j \mod n, j'+i \mod n)$, and $\infty\bowtie(i,j) \circ \infty\bowtie(i',j') = \bowtie(i'+j \mod n, j'+i \mod n)$. Of course composition is associative. As an identity element we have $\bowtie(0,0)$, because it is equal to $Z_{I,I}$ where I is the identity permutation. The inverse element of $\bowtie(i,j)$ is $\bowtie(-i,-j)$, and the inverse of $\infty\bowtie(i,j)$ is $\infty\bowtie(-j,-i)$.

The group presentation is as follows: $\langle x, y, z; x^n = e, y^n = e, z^2 = e, xy = yx, xz = zy, yz = zx\rangle$, with $\bowtie(1,0)$, $\bowtie(0,1)$ and ∞ as generators. This group is a generalization of D_4, which we obtained for the case $n = 2$ in theorem 9. We can see this as a group acting on two planes. The operation ∞ maps the upper plane to the lower plane and vice versa. The operation $\bowtie(1,0)$ can be seem as a translation in the x direction on the lower plane, and in the y direction on the upper plane, while $\bowtie(0,1)$ is a translation in the x direction on the lower plane, and in the y direction on the upper plane.

6 Discussion and further work

In this paper, we presented Lambek–Grishin calculus for n-ary connectives $\mathbf{LG}^{\emptyset}_{\mathbf{n}} + \mathbf{IV}$. This calculus can be seen as a generalization over binary and unary Lambek–Grishin calculus ($\mathbf{LG}^{\emptyset}_{\mathbf{2}} + \mathbf{IV}$ and $\mathbf{LG}^{\emptyset}_{\mathbf{1}} + \mathbf{IV}$)

and is therefore at least mildly context-sensitive. In this paper, we proved that there is a cut-free presentation of the calculus, giving us decidability. We also investigated the symmetries of $\mathbf{LG_n^\emptyset} + \mathbf{IV}$ by making use of group theory.

The extension of Lambek–Grishin calculus to n-ary arity gives rise to many new questions. First, although we know $\mathbf{LG_n^\emptyset} + \mathbf{IV}$ generates non-context-free languages, no upper bound on the generative complexity is known. Secondly, it will be interesting to study the Kripke semantics of the new calculus: it is not yet known whether the results from [7] for $\mathbf{LG_2^\emptyset} + \mathbf{IV}$ and [3] for $\mathbf{LG_1^\emptyset} + \mathbf{IV}$ can be extended to $\mathbf{LG_n^\emptyset} + \mathbf{IV}$. Furthermore, it has been shown that each postulate in the binary calculus can be presented in six interderivable forms [4]. However, in the unary calculus only one form exists for each postulate. It is unknown how the number of interderivable postulates relates to the arity of the connectives. Furthermore, it is useful to try to extend the continuation semantics from Bernardi en Moortgat [1] to the n-ary calculus. Finally, in this paper, we only considered languages with connectives of equal arity. It also will be interesting to see what happens when we combine connectives of different arities.

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